

DSIAC TECHNICAL INQUIRY (TI) RESPONSE REPORT

What Are Fractal Radars, and How Do They Work?

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A chief service of the DoD IACs is free technical inquiry (TI) research, limited to 4 research hours per inquiry. This TI response report summarizes the research findings of one such inquiry jointly conducted by DSIAC.

Abstract

This technical inquiry response provides detail on how fractal radars work. Without knowing the inquirer's knowledge of radars, an assumption was made to start at the beginning with legacy, single-element, radar apertures and show how technology has matured to allow for the use of fractal geometries on aperture surfaces. Fractal equations for radar applications are highly proprietary, so fundamental fractal patterns are used for demonstration purposes in this report. Fractal radar antennas allow for tight coupling of elements with minimized or eliminated sidelobes while maximizing the main beam's power and manifest as identical illumination spots at range, regardless of frequency. Fractal radar's optimized performance can be attributed to the organic efficiencies of fractal patterns in nature, such as in flowers and the flight of birds of prey.

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1.0 TI Request

1.1 Inquiry

What are fractal radars, and how do they work?

1.2 Description

The inquirer would like to broaden her knowledge set on varying emergent technologies regarding radar types, like quantum radars and fractal radars.

2.0 TI Response

Fractal radars earned their name because of the dynamic, fractal patterns that are rendered on a specialized antenna/radar surface (aka antenna topology) to achieve optimization in wideband radar applications. However, there is a whole other science behind fractal radars, which uses fractal equations to perform digital signal processing (DSP) on collected radar images to similarly distinguish, for example, man-made objects from natural ones. This science has been explored abundantly and has little to do with the antenna topology having a fractal pattern on its façade and more to do with the collected, digitized data that is ready for post-collection processing.

Another well-established application of fractals for radar is in image compression. The reader can easily imagine why this application would be beneficial for radar, especially remote antennas. This report focuses more on the topology of fractal radars rather than fractal equations to perform DSP. If the DSP explanation is desired, two sources are provided in the bibliography. This report provides an explanation of fractal radars to provide an elementary explanation for a lay person to understand, not to provide a heavy mathematical or empirically modeled explanation to the advanced technical community.

2.1 Introduction

To begin the discussion, fractal radars are not emergent technologies. Discoveries were made in the late 1990s, and modern commercial-off-the-shelf (COTS) fractal antennas have been available for some time. One provider is Fractal Antenna Systems, Inc., out of Bedford, MA. Fractal Antenna Systems produces COTS fractal antennas like the ones shown in Figures 1 and 2.



Figure 1. Fractal Antenna Systems' MANTA 4-BP [1].

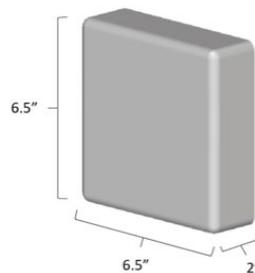


Figure 2. Dimensions of a MANTA 4-BP [1].

While nothing can be gleaned about how a fractal radar works from seeing a picture of a fractal radar antenna, these images are combined with an elementary explanation in this report to broaden the reader's knowledge set. The chief scientist for Fractal Antenna Systems Dr. Nathan (Chip) Cohen was consulted for background on this report.

The remainder of this report splits into two areas: (1) a fundamental explanation of radars (independent of fractals) and (2) fractals. The first topic is short, as much has been published on this subject and it is assumed the reader knows how radars basically work.

2.2 Radar Fundamentals

All radars transmit energy by emitting radiation and receiving reflections, subsequently focusing that received energy to perform postprocessing. Radar systems have evolved considerably, incorporating major advances in material processing; computation and signal processing; and the need for optimized size, weight, and power to make radars easy to transport, set up, operate, and fix in the field. Additionally, the surfaces (i.e., shapes, curvatures, materials) of radar antennas have evolved from a single aperture to multielement arrays. Chunking up radar surfaces into elements has enhanced the applications and accuracy of radars considerably and provided for much smaller apertures (e.g., synthetic-aperture radar arrays). These days,

technology provides for a lot of control on how elements are turned on and what frequencies they should emit, thanks to the electronics on the back end. For example, in active electronically scanned-array radars, the elements are energized electronically, rather than mechanically, to steer antenna beams. It is this technological maturity that enables fractal radars to exist.

Fractal radars operate the same as regular radars in that they still take energy from the environment and focus it for postprocessing. Fractal radars differ from basic radars by employing instantaneously changing topologies that trace out fractal geometric patterns. Any fractal pattern contains segments or edges like the ones that trace out the Koch snowflake shown in Figure 3. While Koch snowflakes are not used for fractal antennas, this graphic illustrates what is meant by segments or edges. The segments of the fractal pattern dictate the shape (i.e., size and location) of each element of a fractal antenna at any given moment. Segment is used carefully here because fractal antenna elements are no longer manifested materially (like in a phased-array radar) but rather electronically. All of this is explained in more detail later in this report.

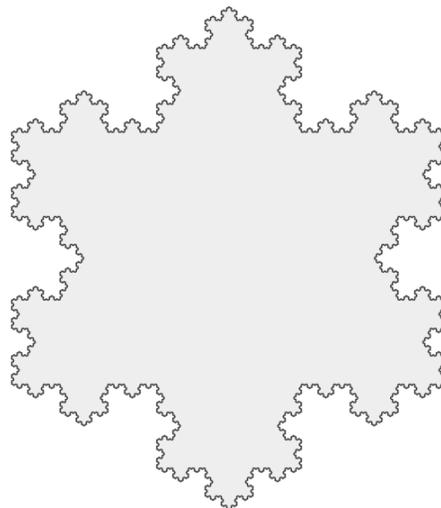


Figure 3. Segments or Edges of a Typical, Generic Fractal Pattern [2].

2.3 Fractals

Fractal patterns are unfurled in nature, and many organisms leverage their fractal shapes to optimize energy collection or energy dispersal. A simple and worthy example found in nature is the flower, which both gives and receives energy. It needs to collect rainwater and sunshine, so

its petal structure is optimized to do that. As the rainwater strikes the petals, it forces the petals to engage in a pattern dance to fold the water over the petals for prime collection. The petals bend toward the sun throughout the day in a dutiful pattern to maximize collection of the sun's rays. The flower also needs to provide pollen, so its pistil is dutifully organized to make extracting pollen easy for bees. One need only replicate the intent of the flower's patterns to optimize radar operations.

Fractal geometries have been previously defined by scientists around the world. Some more famous fractal patterns that have been mathematically defined and geometrically modeled are the previously mentioned Koch snowflake; the Sierpiński triangle; the Julia set; and, purportedly the most famous (due to its inherent beauty), the Mandelbrot set (Figure 4).

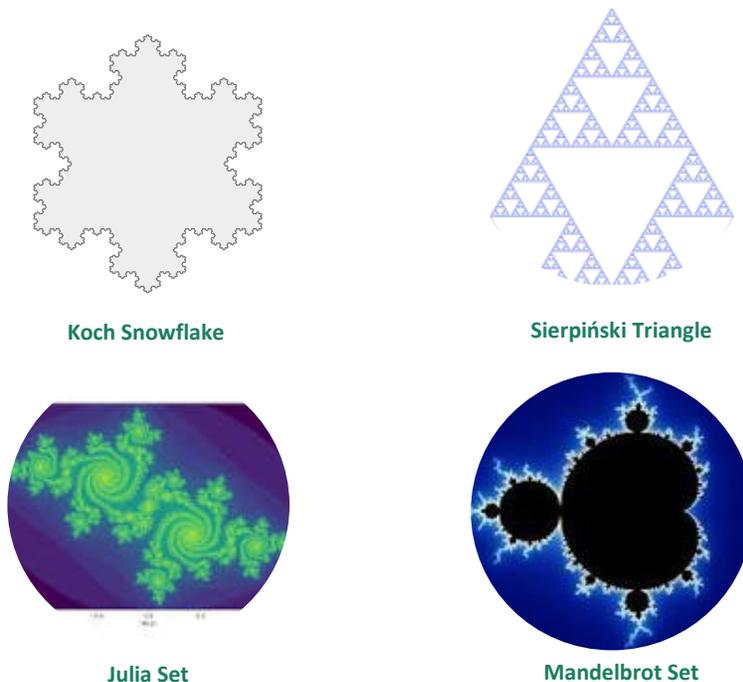


Figure 4. Mathematically Defined and Geometrically Modeled Fractal Patterns [2–5].

While these fractal patterns are heavily studied, the fractal patterns for military and cellular radar applications (such as wideband) do not come from any of these popular shapes. They come from highly coveted and protected mathematical equations that are patented or patent pending for companies like Fractal Antenna Systems. One cannot Google “fractal equations for wideband applications” and get anything useful. The science and application of these proprietary fractal patterns will likely remain shrouded for many decades.

2.4 Fractal Radars

There are two primary benefits of fractal geometries for radar surfaces: (1) minimized sidelobes and (2) optimized power gain per element. Before launching into an explanation of why this is so, it is necessary to define the key structure of a typical radar—the array—and explain what is limiting about complete-aperture arrays.

A continuous or complete array is when the spatial frequencies of the array are continuous. A good example is a legacy radar paraboloid, as shown in Figure 5.



Figure 5. Radar Paraboloid U.S. Army C-Band Radar Dish Antenna [6].

Continuous arrays produce a beam that has sidelobes. When the application is wideband (many frequencies), the sidelobes become a major problem. In radar, they become a particular problem because the radar array is generating sampling in distant space or ranging on items where the beam is not being directed. Naturally, this is a waste of power, which compromises the ability of the main lobe to radar range on the target of interest.

Unlike these legacy, single-aperture, complete antenna arrays, modern radar antenna arrays utilize individual elements. A sparse array is when the spatial frequency spectrum is not

continuous. A phased-array radar like precision acquisition vehicle entry (PAVE) phased-array warning system (PAWS) is a good example of a finite, sparse-aperture antenna array (see Figures 6 and 7). The finite piece indicates the elements are physically molded into the antenna surface. They can be energized dynamically in random or calculated patterns.



Figure 6. PAVE PAWS Array [7].

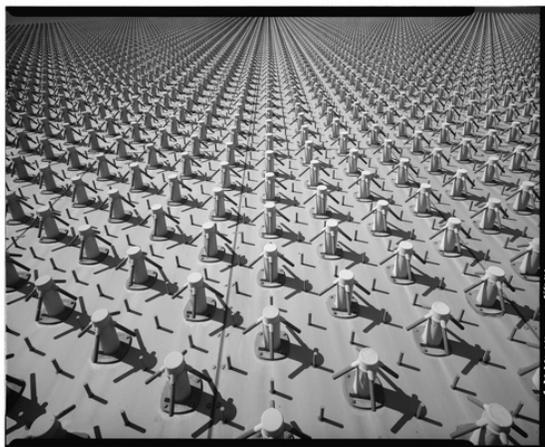


Figure 7. Surface Elements of PAVE PAWS Array [8].

It has been shown for narrow-band applications, when the radar antenna is arranged as a finite, sparse aperture and the elements are energized randomly, the sidelobes are minimized. However, random patterns of elements are not ideal for wideband applications. It turns out fractal distributions are. Getting back to the flower example, when the petals collect sunshine, they are attempting to collect all the light—all the frequencies. Their fractal patterns are optimized for this multiband collection. Similarly, the fractal topologies of a radar antenna are

optimized for transmission and collection of many signals of interest, minimizing sidelobes, thus also optimizing power use.

Due to technological advances in machining antenna surfaces using microfilms and other materials for sensing radio energy, array elements can now be generated on demand and the size and shape of these elements is far more agile than with finite arrays like PAVE PAWS. This is the basis for fractal radars, and now perhaps the images of the fractal antenna systems' MANTA 4-BP antenna should make more sense. Elements are created dynamically by stepping through a fractal geometric algorithm. Unlike the physical elements in Figure 7, in a fractal antenna, one cannot see the elements. The elements are highly dynamic, being energized (turned on) and turned off instantaneously according to a complex fractal calculation. Fractal elements allow for wider band/multiband use with lower mutual coupling between elements, which means they can be placed closer together with fewer sidelobe problems for the array. In addition, the fractal distribution of fractal elements allows near elimination of sidelobes and the same main beam performance at a wideband or at multiple bands. In other words, the spot of illumination is the same, even as frequencies change with the radar. It is almost as if the fractal radar is acting like a multiband laser with a uniform spot size, regardless of frequency.

According to Dr. Cohen, the best way to describe the surface of a fractal antenna array is with the legacy child's toy LiteBrite™ where there is a fixed grid of locations for light pegs but the use of the location for a light peg, the color, and the pattern is variable. Once a fractal equation is programmed into an algorithm, the array is subjected to that calculation and the algorithm drives the on/off states of the light bulbs (or elements) instantaneously. The emitting area of an element at any given time (or with any programmed frequency) can be changed in space, unlike the elements of a finite array like the phased-array radar mentioned earlier. Furthermore, since the elements in a fractal array are generated electronically with photosensitive materials, the speed at which the elements can be energized is faster and the efficiency of the emitters is greater than with mechanical elements.

The fractal algorithm is the key to achieving this optimal radar pattern. It is a geometric function programmed into software and is commensurately dependent on both frequency and time. "These...patterns are the result of [an] equation or mathematical statement. Fractals are created by repeating this equation through a feedback loop in a process called iteration, where the results of one iteration form the input value for the next" [9]. The ability of a fractal radar algorithm to step through its many calculations instantaneously is made possible by modern computer processors (superfast) and sophisticated software programs. It turns out, as the

algorithm is stepped through frequencies, the main beam stays focused and the sidelobes remain minimized. This is a profound and unique benefit of fractal radars, so it bears repeating that the shape of the beam is independent of the frequencies being radiated and the sidelobes remain minimized.

A synopsis of the use of fractal algorithms for antennas/radars can be found in “Fractals in Antennas and Metamaterials Applications:”

...using the fractal geometries [for the shape of] antenna elements. In recent years, there are many fractal shapes that have been proposed for such applications and the designed antennas have significantly improved antenna features such as smaller size, operating in multifrequency bands, with improved power gain and efficiency [10].

What follows is a graphic conceptualization (not a formal model) of how a notional fractal pattern could be generated on a radar aperture. As stated earlier, the actual fractal patterns being used for applications like wideband are highly proprietary, so this conceptual depiction uses a Koch snowflake (Figures 8 and 9) for demonstration purposes only. Incidentally, Koch snowflakes are not the sort of fractal patterns that are used for wideband fractal radar applications.

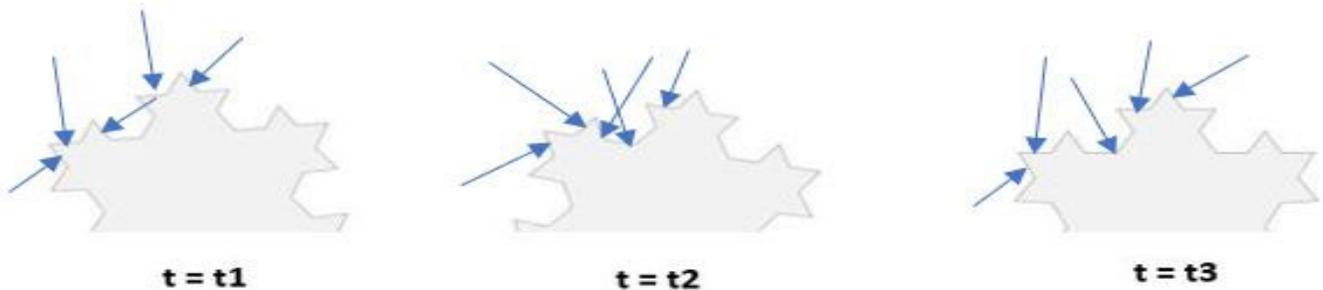


Figure 8. Fractal Geometric Pattern Being Stepped Through Time [11].

The reader will need to engage the imagination to glean the meaning intended for this conceptualization. The surface of a fractal radar is subjected to a continuous algorithmic calculation of the fractal equation, thus stepping the surface through a dynamic pattern of elements or edges in time. The radar echo’s energy is received; collected; and, subsequently, aggregated for the desired application. This same depiction can be used to imagine transmission (i.e., the energy vectors are reversed). Again, the highly dynamic patterns of edges/elements force the energy waves to superimpose or cancel each other optimally to

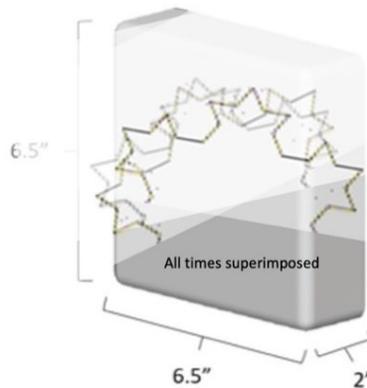


Figure 9. Conceptualization of Fractal Pattern Being Energized on the Surface of a MANTA 4-BP Antenna Aperture (Source: M. Tourangeau).

achieve the desired effect. It is difficult to sketch such a stepping through of a fractal pattern for changing frequencies, so the reader will need to visualize the changing fractal pattern with frequency.

One last word about fractal patterns for radiating applications is, in nature, there exists a special type of spiraling logarithmic pattern called the Fibonacci sequence or Fibonacci series. This pattern starts at zero (in the center) and continues with each subsequent number of elements calculated as the sum of the two previous numbers of elements in the sequence. It is also called the “Golden Ratio.” A common example of this in nature is the center of a sunflower. Another example is the flight of a bird of prey. This spiral pattern is considered nature’s way of being most efficient. The spiral path of a predatory bird of prey is executed with the least amount of loss in energy expenditure and with the most chance of finding and securing its prey. It is not too far a stretch to imagine a dynamically programmed spiral array of elements tracing this series being equally most efficient to collect or radiate radar energy. It certainly would explain the minimization of sidelobes and the optimization of power usage. If manifesting radar antenna aperture arrays is no longer limited by technology, then utilizing time-proven patterns like the Fibonacci series is a prudent approach to realizing desired radar applications.

3.0 Summary

Fractal radars leverage modern computer and material technology to fashion small antenna apertures that are electronically energized to follow a fractal geometric algorithmic pattern to instantaneously create elements that optimize gain and minimize sidelobes in wideband applications. The small form factor, efficient power use, and multiband operation make fractal

radars ideal for multifunction electronic warfare and electromagnetic spectrum operation applications in the field. The algorithms are highly complex and proprietary, so one cannot see what fractal patterns are being used for fractal antennas. However, one can conceptualize the function of a fractal antenna by substituting a more commonly known pattern, such as a Koch snowflake. The best way to think about fractal radars is to imagine the legacy child's toy LiteBrite™. Fractal patterns occur organically in nature and are designed to efficiently collect life-giving energy, such as sunlight and rainwater, or transmit energy, like pollen. These unique, repeating, recursive, inherently efficient patterns make them perfect for reflecting and collecting energy in commonly used radar bands for modern applications.

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Biography

Melinda Tourangeau is the executive director of the RVJ Institute, a 501(c)(3) center of excellence and research institute dedicated exclusively to excellence in the electromagnetic environment. She possesses advanced degrees in electrical engineering and business administration and is currently pursuing a Ph.D. in Education with an emphasis on organizational systems. Ms. Tourangeau is considered a subject matter expert in electromagnetic warfare (EW) and electromagnetic spectrum operations. She has authored numerous reports for the U.S. Department of Defense (DoD) and U.S. Congress and given presentations to audiences in Europe, Hawaii, Canada, and the United States. Her background emphasizes electro-optics, lasers, and semiconductor physics, as well as organizational and leadership systems. Her career includes serving as a DoD program manager for critical EW programs and serving as a spectrum engineer for Verizon Wireless.

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